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Flat Points of Minimal Balance Surfaces

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Abstract

The symmetry conditions for flat points of minimal surfaces have been studied in relation to the order β of points on such surfaces. Using symmetry aspects, a set of rules for the derivation of flat points have been developed. By means of these rules the flat points for the 45 families of minimal balance surfaces known so far have been determined. As a check for completeness the relation between the genus of a minimal surface and the orders of its flat points has been used.

1. Introduction

A minimal surface in three-dimensional space is a surface that fulfils the condition

$$k_1 + k_2 = 0$$

for each of its points. k_1 and k_2 are the main curvatures of the surface. For most points this defining condition holds such that

$$k_1 = -k_2 \neq 0.$$

Then the surrounding area of the point has a saddle-like shape. For exceptional points, however,

$$k_1 = k_2 = 0$$

is fulfilled. Such points are called the *flat points* of the surface.

In the surrounding of a flat point the surface shows $j > 2$ valleys which are separated by j ridges. If a tiling on the minimal surface is constructed in such a way that all flat points lie on vertices and the edges are defined by lines of curvature, more than four (at least six) tiles meet at each flat point. The best known example of a flat point is the 'monkey saddle' with $j = 3$. It has already been observed for the classical three-periodic minimal surfaces of Schwarz (1890).

For two reasons, the flat points on three-periodic minimal surfaces are of special interest.

(1) They may be used for the parametrization of the surfaces (*cf.* Lidin & Hyde, 1987).

(2) There exists a relationship between the flat points of a surface and its genus (*cf.* Hyde, 1989; Hopf, 1983).

The latter property has been used to derive a complete list of flat points of all minimal balance surfaces known so far (Fischer & Koch 1987, 1989*a*, *b*; Koch & Fischer, 1988, 1989*a*, *b*). In addition, the relation between the order of a flat point and its site symmetry has been studied for this purpose.

2. Order and site symmetry of flat points

For any point of an intersection-free minimal surface [which need not be a minimal balance surface, *cf.* Fischer & Koch (1987)], the degree of its flatness may be characterized by a non-negative integer β, called its *order*.

Let P_0 be a point of a minimal surface and $\mathbf{n}_0$ the normal vector at that point. Then the order of P_0 may be derived as follows (*cf.* Hyde, 1989). A second point P is moved on the surface around P_0 and the behaviour of its normal vector $\mathbf{n}$ during this motion is considered. If P_0 is not a flat point but an ordinary point, $\mathbf{n}$ rotates once around $\mathbf{n}_0$ during one revolution of P around P_0 (*cf.* Fig. 1). If, however, P_0 is a flat point, $\mathbf{n}$ rotates more than once around $\mathbf{n}_0$ per revolution of P, say p times. Then the order β of P_0 is defined as

$$\beta = p - 1.$$

Accordingly, an ordinary point has order $\beta = 0$, whereas the order of a flat point may be any positive integer. As so far only orders up to $\beta = 4$ have been observed for three-periodic minimal surfaces, the geometrical situation is illustrated in Figs. 2 to 5 for flat points with $\beta = 1$ (monkey saddle), 2, 3 and 4, respectively.

As may be learned from these figures, the number j of valleys (or of ridges) surrounding a flat point is

$$j = \beta + 2.$$

The figures display, in addition, the maximal site symmetry compatible with a (flat) point of given order. This maximal site symmetry is $\bar{4}m2$ for $\beta = 0$ (ordinary point), $\bar{3}m$ for $\beta = 1$ (monkey saddle), $\bar{8}m2$ for $\beta = 2$, $\bar{5}m$ for $\beta = 3$, and $\overline{12}m2$ for $\beta = 4$. The relation between β and the maximal site symmetry can more easily be expressed if rotoreflections instead of rotoinversions are considered. Then the maximal site symmetry is $\tilde{N}m2$ or $\tilde{N}m$ for β even or odd, respectively, with $N = 2j = 2\beta + 4$.

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For three-periodic minimal surfaces this maximal site symmetry can only be realized for $\beta = 0$ and $\beta = 1$, but any corresponding subgroup may also occur as site symmetry of a (flat) point on a minimal surface. Flat points with any higher value of β and maximal site symmetry, however, occur, for example, on a special kind of one-periodic minimal surface, called saddle towers (*cf.* Karcher, 1988).

The relations between the orders of (flat) points and their possible site symmetries are summarized in a subgroup diagram (Fig. 6). In analogy with the symmetry description of minimal balance surfaces introduced by Fischer & Koch (1987), group-subgroup pairs of point groups are used in this diagram, if the site symmetry contains symmetry operations that interchange the two sides of the surface. In this case the second symbol refers to that subgroup of index 2 that does not interchange the sides. If, however, all site-symmetry operations preserve each side of the surface, only one symbol is given. For points on minimal surfaces that subdivide the three-dimensional space into two non-congruent regions only the latter case may occur. The orientation of the site-symmetry elements with respect to the minimal surface is unambiguous except for one case (*cf.* Fischer & Koch, 1987; Koch & Fischer 1988). With the exception of twofold axes all rotation and rotoinversion axes and all mirror planes have to be perpendicular to the surface. For site symmetry 2, however, an axis $2_\perp$ perpendicular to the surface has to be distinguished from an axis $2_\parallel$ within the surface.

As Fig. 6 shows, most site symmetries of points on intersection-free minimal surfaces necessarily enforce these points to be flat points with some minimal orders. Conversely, only points with site symmetry $\bar{4}m2$-222, $\bar{4}$-2, 222-2, $2mm$, 2, m or 1 can be ordinary points on minimal balance surfaces and only those with site symmetry $2mm$, 2, m or 1 can be such points on other intersection-free three-periodic minimal surfaces. It should be noticed, however, that the diagram is incomplete: for site symmetry 3, for example, β

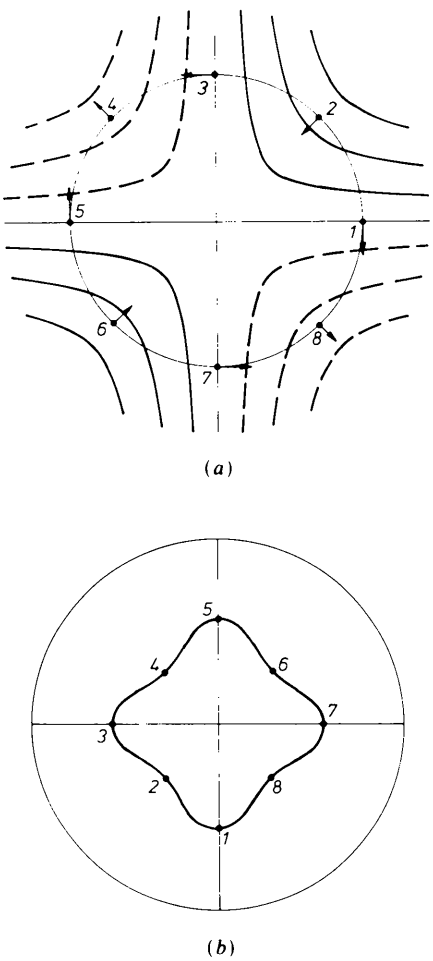

Fig. 1. (a) Surrounding of a (flat) point with order β and maximal site symmetry. The arrows represent projections of the normal vectors of the surface. (b) The change of the direction of the normal vector along the closed path indicated in (a) is illustrated in a stereographic projection. $\beta = 0$: site symmetry $\bar{4}m2$-$2mm$, ordinary point.

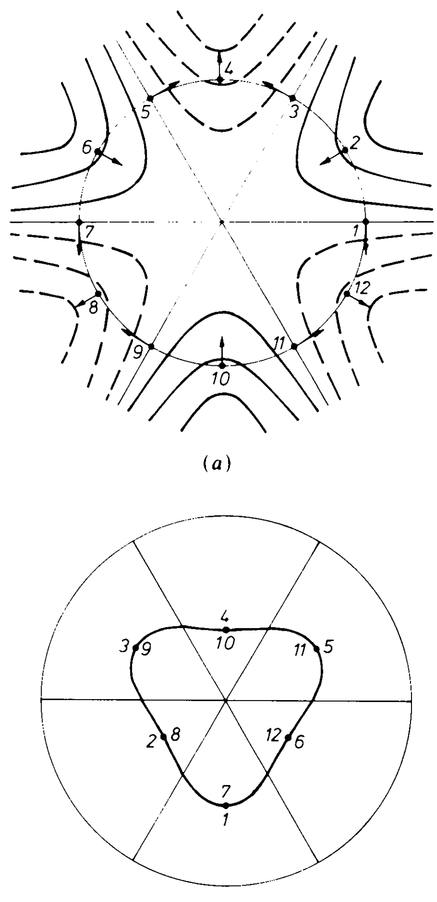

Fig. 2. As Fig. 1. $\beta = 1$: site symmetry $\bar{3}m$-$3m$, monkey saddle.

may take not only the values 1 and 4, but any value $j-2$ with j any positive integer divisible by 3, because the number of valleys or ridges must be divisible by 3. In the derivation of flat points described below, however, such higher values of β never had to be considered.

3. Flat points and genera of minimal surfaces

For the decision whether all the flat points of an intersection-free three-periodic minimal surface have been found, the following equation going back to Hopf (1983) and having been introduced by Hyde (1989) may be used:

$$g = 1 + \tfrac{1}{4}\sum \beta_i.$$

Here, g is the genus of the surface defined with respect to a primitive unit cell of the oriented surface (*cf.* Fischer & Koch, 1989*c*) and the summation runs over all flat points within this unit cell. For minimal balance surfaces this means the following: If G is the full symmetry of the surface and H is its subgroup of index 2 not interchanging both sides of the surface (Fischer & Koch, 1987), then the summation has to be carried out over a primitive unit cell of H.

For practical reasons, however, it is more convenient to refer to a primitive unit cell of G and to modify the formula such that only symmetrically inequivalent flat points are considered. For this, two cases have to be distinguished:

(*a*) If H is a translation-equivalent subgroup of G, *i.e.* no translation of G interchanges the two sides of the surface, then

$$g = 1 + \tfrac{1}{4}\sum m_i\beta_i \tag{1}$$

holds

(*b*) If H is a class-equivalent subgroup of G, *i.e.* half of the translations of G interchange the two sides of the surface, the corresponding equation is

$$g = 1 + \tfrac{1}{2}\sum m_i\beta_i. \tag{2}$$

In both cases i runs over all kinds of symmetrically inequivalent flat points and m_i means the multiplicity of the ith kind referred to a primitive unit cell of G.

As g is at least 3 for any intersection-free three-periodic minimal surface (*cf.* Fischer & Koch, 1989*c*),

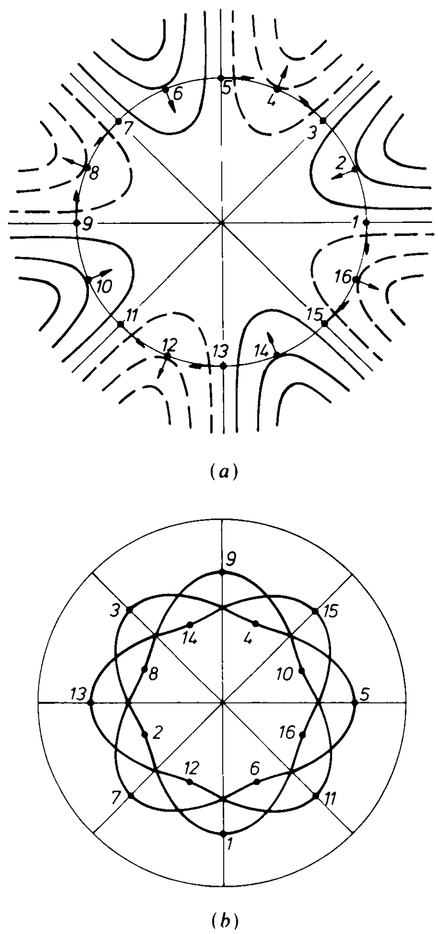

Fig. 3. As Fig. 1. $\beta = 2$: site symmetry $\bar{8}m2$-$4mm$.

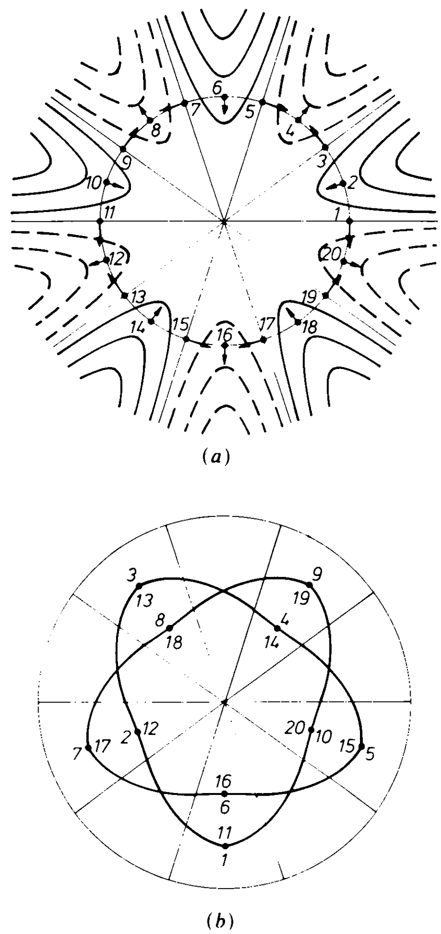

Fig. 4. As Fig. 1. $\beta = 3$: site symmetry $\bar{5}m$-$5m$.

it follows from (1) and (2) that each such surface must contain flat points.

4. Derivation of flat points

The symmetry conditions described above together with some general considerations have been used for the complete derivation of all flat points of the known minimal balance surfaces. The procedure for this is composed of several successive steps each followed by a check on completeness by means of (1) or (2).

(*a*) Each point with site symmetry 622-6, 422-4, 32-3, $\bar{3}m$-$3m$, $\bar{3}$-3, $2/m$-m, or $\bar{1}$-1 that lies on the surface must be a flat point with some minimal order that may be read from Fig. 6. If the inherent symmetry of the minimal balance surface is G-H, the Wyckoff positions of these points may be taken from Table 1 of Koch & Fischer (1988). In most such cases not only the locations of the flat points but also their normals are fixed by symmetry. P surfaces (Schwarz, 1890) with symmetry $Im\bar{3}m$-$Pm\bar{3}m$, for example, must contain flat points with $\beta = 1$ (at least) at Wyckoff position $Im\bar{3}m$ 8(c) . $\bar{3}m$ $\frac{1}{4}\frac{1}{4}\frac{1}{4}$ and with directions $\langle 111 \rangle$ of the normals. As the genus is 3 (*cf.* Fischer & Koch, 1989*c*), (2) proves that the surface has no additional flat points. For site symmetry $2/m$-2 the normal at the flat point is only fixed to lie within the mirror plane. In the case of site symmetry $\bar{1}$-1 its direction is not restricted at all.

(*b*) If a six-, four- or threefold rotation axis crosses the minimal (balance) surface, the intersection point also has to be a flat point. Its site symmetry then is $6mm$, 6, $4mm$, 4, $3m$ or 3 with respect to G as well as to H. In this case the normal coincides with the rotation axis, but the location of the flat point is not further fixed by symmetry. The respective coordinate parameter may be estimated, *e.g.* from a model of the surface, or it may be calculated, if the parametrization of the surface is known. A $C(P)$ surface (Neovius, 1883), for example, has the same symmetry as a P surface, but its genus is 9. Therefore, it must possess additional flat points. As the symmetry of a generating circuit of a $C(P)$ surface is $4m.m$ (*cf.* Fischer & Koch, 1987), the remaining flat points have $\beta = 2$ and are located at $Im\bar{3}m$ 12(e) $4m.m$ $x00$ with normals parallel to $\langle 100 \rangle$.

(*c*) If a minimal balance surface contains twofold axes, the behaviour of the normal when moving along such an axis may give conclusive indications of flat points. This is the case if the rotation of the normal (within the plane perpendicular to the twofold axis) changes its sense. Then, surrounding the point where the reversal takes place there exists for each point on the twofold axis a second opposing point on that axis with the same direction of the normal. As a consequence the reversal point must be a flat point of odd order. Such flat points are neither fixed with respect to their location nor with respect to their normal direction (within the plane perpendicular to the twofold axis). In particular, if an edge of a generating circuit ends in two vertices with parallel normals, a flat point has to lie on this edge. In the case of catenoid-like surface patches (*cf.* Koch & Fischer,

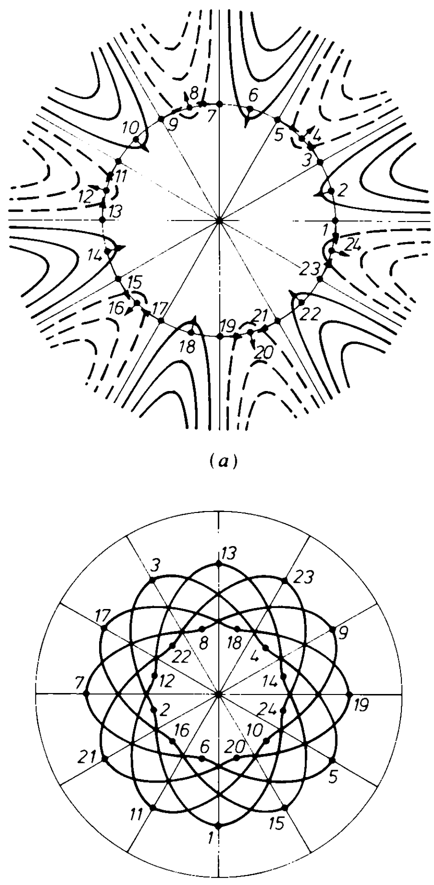

Fig. 5. As Fig. 1. $\beta = 4$: site symmetry $\overline{12}m2$-$6mm$.

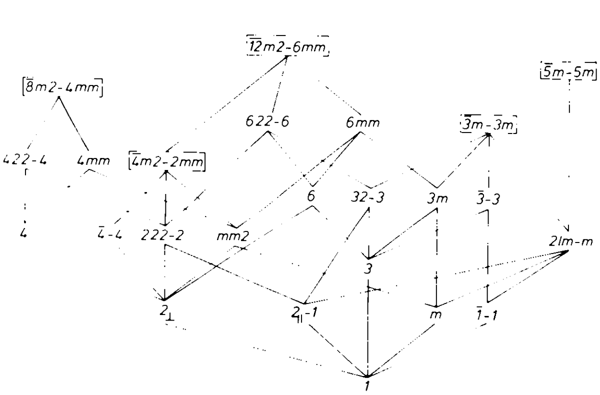

Fig. 6. Group–subgroup diagram representing the maximal and the possible crystallographic site symmetries of (flat) points with order up to 4.

1988), each edge, therefore, contains at least one flat point of order one. Y and $C(Y)$ surfaces (Fischer & Koch, 1987), for example, have the same symmetry $I4_132$-$P4_332$ and the same linear skeletal net [$I4_132$ $24(f)$ $2\,.\,.$ $x0\frac{1}{4}$ and $24(g)$ $.\,.\,2$ $\frac{1}{8}, y, \frac{1}{4}+y$] with skew polygons as generating circuits. According to (a), both surfaces must contain flat points at $8(b)$ $.32$ $\frac{7}{8}\frac{7}{8}\frac{7}{8}$. According to ($b$), $C(Y)$ surfaces show additional ones at $16(e)$ $.3.$ xxx. If the orders of all these flat points are assumed to be $\beta = 1$, additional such flat points within Wyckoff positions of $I4_132$ with multiplicity 24, *i.e.* on twofold axes, are needed in both cases. Inspection of the generating circuits shows that these flat points lie on $24(g)$ in the case of a Y, but on $24(f)$ in the case of a $C(Y)$ surface.

(d) If a minimal surface intersects a mirror plane in a line with an inflection point, this point must be a flat point of odd order because the intersection line is a plane line of curvature and its curvature at the inflection point is zero. Such a flat point and its normal are only fixed to lie within the mirror plane. Each multiple catenoid (Koch & Fischer, 1989a), for example, is halved by a mirror plane producing an intersection line with inflection points.

(e) In the case of surface patches which are catenoids with spout-like attachments (Koch & Fischer, 1989b), the points where the spouts are united to three- or four-armed handles are flat points of order $\beta = 1$ or $\beta = 2$, respectively. In most cases, however, these points are distinguished by symmetry and rule (b) or (d) can be applied.

5. Discussion of the results

For all but two families of minimal balance surfaces, the Wyckoff positions and the orders of their flat points could be derived unambiguously. In addition, the flat-point coordinates could be determined at least approximately. If the location of a flat point was not entirely fixed by symmetry, it was estimated from a model.*

The results are compiled in Table 1. The directions of the flat-point normals are not tabulated. If they are fixed by symmetry they can be read from the site-symmetry symbols. If a normal is only restricted to a plane, its free parameter may depend on the axial ratio [*cf.* the study of *CLP* surfaces by Lidin & Hyde (1987)]. Then either the direction of the normal or the axial ratio may arbitrarily be chosen. In other cases, however, the direction parameter has a specific but unknown value [*e.g.* for the flat points of Y and $C(Y)$ surfaces discussed above]. In even more complicated cases several parameters may be related to each other in an unknown way and all of them may depend on the axial ratio(s). This happens, for example, for the two kinds of symmetrically independent flat points of *HS*3 surfaces (Koch & Fischer, 1988).

In total Table 1 contains information on 45 families of minimal balance surfaces. For 13 of these, *i.e.* *P*, *D*, *CLP*, *tD*, *oCLP*, *oDa*, *oDb*, *H*, *tP*, *oPb*, *rPD*, Y^* [the gyroid surface of Schoen (1970)] and *oPa* surfaces, rule (a) was sufficient to find all flat points. Rule (b) had to be used for $C(P)$† and $C(D)$ surfaces. For ten additional families, *i.e.* $C(S)$, S, Y, $C(Y)$, *HS*1, *HS*2, *R*3, *R*2, *HS*3 and $C(^{\pm}Y)$ surfaces, rule (c) had to be applied in addition. These three rules suffice to determine the flat points for all minimal balance surfaces with disc-like or catenoid-like surface patches (Fischer & Koch, 1987; Koch & Fischer, 1988) and for three additional ones. Rule (d) helps to locate the flat points for the eight families with multiple catenoids as surface patches (Koch & Fischer, 1989a) and of four families made up from catenoids with spout-like attachments, namely $C(H)$, $tC(P)$,† $C(R2)$ and *PT* surfaces (Koch & Fischer, 1989b). For the remaining two families of the latter kind, *i.e.* $C(R3)$ and $oC(P)$† surfaces, the list of flat points can only be completed by means of rule (e). In all these 39 cases there is no doubt about the Wyckoff positions of the flat points and about their orders which always take the lowest values compatible with the site symmetries.

The situation is more complicated for the remaining six types, *i.e.* for all surfaces built up from branched catenoids or from infinite strips and for $^{\pm}Y$ surfaces (Fischer & Koch, 1989a, b, 1987, respectively): *BC*1, *BC*3, *ST*1 and *ST*2 surfaces either must possess flat points with $\beta = 1$ in general position or, instead of this, flat points on the twofold axes, namely one kind with $\beta = 2$ or two kinds with $\beta = 1$. In all four cases, the models clearly show flat points in general position. A similar situation arises for $^{\pm}Y$ surfaces. Here, however, the model indicates flat points in Wyckoff position $Ia\bar{3}$ $24(d)$ $2\,.\,.$ $x0\frac{1}{4}$ with $\beta = 2$ rather than in the general position with $\beta = 1$. *BC*2 surfaces must either have flat points at $P4_2/nnm$ $4(f)$ $.\,.\,2/m$ $\frac{3}{4}\frac{3}{4}\frac{3}{4}$ with $\beta = 3$ or flat points at $4(f)$ and at $8(m)$ $.\,.\,m$ xxz, both with $\beta = 1$. Then, each flat point with $\beta = 3$ corresponds to three closely adjacent flat points with $\beta = 1$. The models of *BC*2 surfaces imply flat points

* To produce small models of surface patches, spring-steel wire was bent in the shape of generating circuits and spanned either temporarily by soap films or permanently by a special lacquer. To obtain models of larger parts of minimal balance surfaces stronger wires were welded to form parts of the linear skeletal nets and material from nylon stockings was used to span the generating circuits. The authors gratefully acknowledge the help of Mr K.-H. Linker, Mr W. Schmidtke, Mr H. Kilian and Mrs A. Senger for producing the wire frames and of Dipl. Min. I. Trautmann and Dipl. Min. A. Fett for part of the sewing.

† The comparison of the flat points of $C(P)$, $tC(P)$ and $oC(P)$ surfaces as listed in Table 1 shows that two flat points with order one and site symmetry m of a $tC(P)$ or an $oC(P)$ surface correspond to one flat point of order 2 of a $C(P)$ surface.

Table 1. *The flat points of the minimal balance surfaces*

Minimal surface	G–H	Genus	Flat points: Order	Wyckoff position	Parameters
P	$Im\bar{3}m-Pm\bar{3}m$	3	1	$8(c)\ .\bar{3}m\ \frac{1}{4}\frac{1}{4}\frac{1}{4}$	
$C(P)$	$Im\bar{3}m-Pm\bar{3}m$	9	2	$12(e)\ 4m.m\ x_1 00$	$x_1 \simeq 0{\cdot}42$
			1	$8(c)\ .\bar{3}m\ \frac{1}{4}\frac{1}{4}\frac{1}{4}$	
D	$Pn\bar{3}m-Fd\bar{3}m$	3	1	$4(c)\ .\bar{3}m\ \frac{3}{4}\frac{3}{4}\frac{3}{4}$	
$C(D)$	$Pn\bar{3}m-Fd\bar{3}m$	19	4	$8(e)\ .3m\ x_1x_1x_1$	$x_1 \simeq 0{\cdot}21$
			1	$4(c)\ .\bar{3}m\ \frac{3}{4}\frac{3}{4}\frac{3}{4}$	
$C(S)$	$Ia\bar{3}d-Ia\bar{3}$	9	1	$16(b)\ .32\ \frac{1}{8}\frac{1}{8}\frac{1}{8}$	
				$48(g)\ ..2\ \frac{1}{8}, y_1, \frac{1}{4}-y_1$	$y_1 \simeq 0{\cdot}38$
S	$Ia\bar{3}d-I\bar{4}3d$	11	1	$16(a)\ .\bar{3}.\ 000$	
				$16(b)\ .32\ \frac{1}{8}\frac{1}{8}\frac{1}{8}$	
				$48(g)\ ..2\ \frac{1}{8}, y_1, \frac{1}{4}-y_1$	$y_1 \simeq 0{\cdot}25$
Y	$I4_132-P4_332$	9	1	$8(b)\ .32\ \frac{7}{8}\frac{7}{8}\frac{7}{8}$	
				$24(g)\ ..2\ \frac{1}{8}, y_1, \frac{1}{4}+y_1$	$y_1 \simeq 0{\cdot}10$
$C(Y)$	$I4_132-P4_332$	13	1	$8(b)\ .32\ \frac{7}{8}\frac{7}{8}\frac{7}{8}$	
				$16(e)\ .3.\ x_1x_1x_1$	$x_1 \simeq 0{\cdot}08$
				$24(f)\ 2..\ x_2 0\frac{1}{4}$	$x_2 \simeq 0{\cdot}28$
$HS1$	$P6_222-P6_122(2c)$	7	1	$6(f)\ 2..\ \frac{1}{2}0z_1$	$z_1 \simeq 0{\cdot}14$
				$6(h)\ .2.\ x_2 0\frac{1}{2}$	$x_2 \simeq 0{\cdot}28$
$HS2$	$P6_222-P3_212$	4	1	$6(f)\ 2..\ \frac{1}{2}0z_1$	$z_1 \simeq 0{\cdot}18$
				$6(h)\ .2.\ x_2 0\frac{1}{2}$	$x_2 \simeq 0{\cdot}20$
CLP	$P4_2/mcm-P4_2/mmc(v)$	3	1	$4(f)\ 2/m..\ 0\frac{1}{2}0$	
tD	$P4_2/nnm-I4_1/amd$	3	1	$4(f)\ ..2/m\ \frac{3}{4}\frac{3}{4}\frac{3}{4}$	
$oCLP$	$Pccm-Cccm$	3	1	$2(c)\ ..2/m\ 0\frac{1}{2}0$	
				$2(d)\ ..2/m\ \frac{1}{2}00$	
oDa	$Pnnn-Fddd$	3	1	$4(f)\ \bar{1}\ \frac{3}{4}\frac{3}{4}\frac{3}{4}$	
oDb	$Cmma-Imma$	3	1	$4(d)\ 2/m..\ 00\frac{1}{2}$	
				$4(e)\ .2/m.\ \frac{1}{4}\frac{1}{4}0$	
H	$P6_3/mmc-P\bar{6}m2$	3	1	$2(a)\ \bar{3}m.\ 000$	
				$6(g)\ .2/m.\ \frac{1}{2}00$	
$MC1$	$P6_3/mcm-P\bar{6}2m$	7	1	$2(b)\ \bar{3}.m\ 000$	
				$4(d)\ 3.2\ \frac{1}{3}\frac{2}{3}0$	
				$6(f)\ ..2/m\ \frac{1}{2}00$	
				$12(j)\ m..\ x_1y_1\frac{1}{4}$	$x_1 \simeq 2y_1 \simeq 0{\cdot}30$
$C(H)$	$P6_3/mmc-P\bar{6}m2$	7	1	$2(a)\ \bar{3}m.\ 000$	
				$4(f)\ 3m.\ \frac{1}{3}\frac{2}{3}z_1$	$z_1 \simeq 0{\cdot}20$
				$6(g)\ .2/m.\ \frac{1}{2}00$	
				$12(j)\ m..\ x_2y_2\frac{1}{4}$	$x_2 \simeq 0{\cdot}20$, $y_2 \simeq 0$
$R3$	$P6/mcc-P6/m$	13	4	$2(a)\ 622\ 00\frac{1}{4}$	
			1	$4(c)\ 3.2\ \frac{1}{3}\frac{2}{3}\frac{1}{4}$	
				$12(j)\ .2.\ x_1 0\frac{1}{4}$	$x_1 \simeq 0{\cdot}33$
				$12(k)\ ..2\ x_2, 2x_2, \frac{1}{4}$	$x_2 \simeq 0{\cdot}20$
				$12(k)\ ..2\ x_3, 2x_3, \frac{1}{4}$	$x_3 \simeq 0{\cdot}43$
$MC2$	$P6/mcc-P6/m$	13	4	$2(a)\ 622\ 00\frac{1}{4}$	
			1	$4(c)\ 3.2\ \frac{1}{3}\frac{2}{3}\frac{1}{4}$	
				$12(k)\ ..2\ x_1, 2x_1, \frac{1}{4}$	$x_1 \simeq 0{\cdot}20$
				$12(l)\ m..\ x_2y_2 0$	$x_2 \simeq 0{\cdot}38$, $y_2 \simeq 0$
				$12(l)\ m..\ x_3y_3 0$	$x_3 \simeq y_3/2 \simeq 0{\cdot}45$
$MC3$	$P6/mcc-P6/m$	13	4	$2(a)\ 622\ 00\frac{1}{4}$	
			1	$4(c)\ 3.2\ \frac{1}{3}\frac{2}{3}\frac{1}{4}$	
				$12(j)\ .2.\ x_1 0\frac{1}{4}$	$x_1 \simeq 0{\cdot}33$
				$12(l)\ m..\ x_2y_2 0$	$x_2 \simeq y_2/2 \simeq 0{\cdot}22$
				$12(l)\ m..\ x_3y_3 0$	$x_3 \simeq y_3/2 \simeq 0{\cdot}42$

Table 1 (*cont.*)

Minimal surface	$G-H$	Genus	Flat points: Order	Wyckoff position	Parameters
$MC4$	$P6/mcc-P6/m$	13	4	$2(a)\ 622\ 00\frac{1}{4}$	
			1	$4(c)\ 3.2\ \frac{1}{3}\frac{2}{3}\frac{1}{4}$	
				$12(k)\ ..2\ x_1, 2x_1, \frac{1}{4}$	$x_1 \simeq 0{\cdot}43$
				$12(l)\ m..\ x_2y_20$	$x_2 \simeq 0{\cdot}27,\ y_2 \simeq 0$
				$12(l)\ m..\ x_3y_30$	$x_3 \simeq y_3/2 \simeq 0{\cdot}16$
$C(R3)$	$P6/mcc-P6/m$	37	4	$2(a)\ 622\ 00\frac{1}{4}$	
			1	$4(c)\ 3.2\ \frac{1}{3}\frac{2}{3}\frac{1}{4}$	
				$12(j)\ .2.\ x_10\frac{1}{4}$	$x_1 \simeq 0{\cdot}33$
				$12(k)\ ..2\ x_2, 2x_2, \frac{1}{4}$	$x_2 \simeq 0{\cdot}20$
				$12(k)\ ..2\ x_3, 2x_3, \frac{1}{4}$	$x_3 \simeq 0{\cdot}43$
				$12(l)\ m..\ x_4y_40$	$x_4 \simeq 0{\cdot}22,\ y_4 \simeq 0$
				$12(l)\ m..\ x_5y_50$	$x_5 \simeq 0{\cdot}44,\ y_5 \simeq 0$
				$12(l)\ m..\ x_6y_60$	$x_6 \simeq y_6/2 \simeq 0{\cdot}14$
				$12(l)\ m..\ x_7y_70$	$x_7 \simeq y_7/2 \simeq 0{\cdot}27$
				$12(l)\ m..\ x_8y_80$	$x_8 \simeq y_8/2 \simeq 0{\cdot}38$
				$12(l)\ m..\ x_9y_90$	$x_9 \simeq y_9/2 \simeq 0{\cdot}46$
				$24(m)\ 1\ x_{10}y_{10}z_{10}$	$x_{10} \simeq 0{\cdot}45,\ y_{10} \simeq 0{\cdot}13,\ z_{10} \simeq 0{\cdot}05$
tP	$I4/mmm-P4/mmm$	3	1	$8(f)\ ..2/m\ \frac{1}{4}\frac{1}{4}\frac{1}{4}$	
$MC5$	$P4_2/mcm-Cmmm$	5	1	$8(m)\ .2.\ x_1\frac{1}{2}\frac{1}{4}$	$x_1 \simeq 0{\cdot}22$
				$8(n)\ m..\ x_2y_20$	$x_2 \simeq 0{\cdot}20,\ y_2 \simeq 0$
$tC(P)$	$I4/mmm-P4/mmm$	9	2	$4(e)\ 4mm\ 00z_1$	$z_1 \simeq 0{\cdot}45$
			1	$8(f)\ ..2/m\ \frac{1}{4}\frac{1}{4}\frac{1}{4}$	
				$16(l)\ m..\ x_2y_20$	$x_2 \simeq 0{\cdot}40,\ y_2 \simeq 0{\cdot}10$
$R2$	$I4/mcm-P4/mbm$	9	2	$4(a)\ 422\ 00\frac{1}{4}$	
			1	$8(e)\ ..2/m\ \frac{1}{4}\frac{1}{4}\frac{1}{4}$	
				$16(j)\ .2.\ x_10\frac{1}{4}$	$x_1 \simeq 0{\cdot}33$
$MC6$	$I4/mcm-P4/mbm$	9	2	$4(a)\ 422\ 00\frac{1}{4}$	
			1	$8(e)\ ..2/m\ \frac{1}{4}\frac{1}{4}\frac{1}{4}$	
				$16(k)\ m..\ x_1y_10$	$x_1 \simeq 0{\cdot}35,\ y_1 \simeq 0$
$MC7$	$P4/mcc-P4/m$	9	2	$2(a)\ 422\ 00\frac{1}{4}$	
				$2(c)\ 422\ \frac{1}{2}\frac{1}{2}\frac{1}{4}$	
			1	$8(l)\ .2.\ x_1\frac{1}{2}\frac{1}{4}$	$x_1 \simeq 0{\cdot}33$
				$8(m)\ m..\ x_2y_20$	$x_2 \simeq 0{\cdot}25,\ y_2 \simeq 0$
				$8(m)\ m..\ x_3y_30$	$x_3 \simeq y_3 \simeq 0{\cdot}22$
$C(R2)$	$I4/mcm-P4/mbm$	25	2	$4(a)\ 422\ 00\frac{1}{4}$	
			1	$8(e)\ ..2/m\ \frac{1}{4}\frac{1}{4}\frac{1}{4}$	
				$16(j)\ .2.\ x_1 0\frac{1}{4}$	$x_1 \simeq 0{\cdot}33$
				$16(k)\ m..\ x_2y_20$	$x_2 \simeq y_2 \simeq 0{\cdot}15$
				$16(k)\ m..\ x_3y_30$	$x_3 \simeq 0{\cdot}20,\ y_3 \simeq 0$
				$16(k)\ m..\ x_4y_40$	$x_4 \simeq 0{\cdot}40,\ y_4 \simeq 0$
				$16(l)\ ..m\ x_5, \frac{1}{2}+x_5, z_5$	$x_5 \simeq 0{\cdot}20,\ z_5 \simeq 0{\cdot}05$
oPb	$Fmmm-Cmmm$	3	1	$8(c)\ 2/m..\ 0\frac{1}{4}\frac{1}{4}$	
				$8(d)\ .2/m.\ \frac{1}{4}0\frac{1}{4}$	
$oMC5$	$Pccm-P2/m$	5	1	$4(j)\ 2..\ x_1\frac{1}{2}\frac{1}{4}$	$x_1 \simeq 0{\cdot}22$
				$4(l)\ .2.\ \frac{1}{2}y_2\frac{1}{4}$	$y_2 \simeq 0{\cdot}22$
				$4(q)\ ..m\ x_3y_30$	$x_3 \simeq 0{\cdot}20,\ y_3 \simeq 0$
				$4(q)\ ..m\ x_4y_40$	$x_4 \simeq 0,\ y_4 \simeq 0{\cdot}20$
$oC(P)$	$Fmmm-Cmmm$	9	2	$8(i)\ mm2\ 00z_1$	$z_1 \simeq 0{\cdot}05$
			1	$8(c)\ 2/m..\ 0\frac{1}{4}\frac{1}{4}$	
				$8(d)\ .2/m.\ \frac{1}{4}0\frac{1}{4}$	
				$16(o)\ ..m\ x_2y_20$	$x_2 \simeq 0{\cdot}25,\ y_2 \simeq 0{\cdot}15$
				$16(o)\ ..m\ x_3y_30$	$x_3 \simeq 0{\cdot}15,\ y_3 \simeq 0{\cdot}25$
PT	$Fmmm-Cmmm$	5	1	$8(c)\ 2/m..\ 0\frac{1}{4}\frac{1}{4}$	
				$8(d)\ .2/m.\ \frac{1}{4}0\frac{1}{4}$	
				$16(o)\ ..m\ x_1y_10$	$x_1 \simeq 0{\cdot}25,\ y_1 \simeq 0{\cdot}10$
$HS3$	$P6_222-P6_422(2c)$	7	1	$6(h)\ .2.\ x_10\frac{1}{2}$	$x_1 \simeq 0{\cdot}15$
				$6(j)\ ..2\ x_2, 2x_2, \frac{1}{2}$	$x_2 \simeq 0{\cdot}20$

Table 1 (*cont.*)

Minimal surface	G–H	Genus	Flat points: Order	Flat points: Wyckoff position	Flat points: Parameters
$ST1$	$P6_222$–$P6_422(2c)$	7	1	$12(k)\ 1\ x_1y_1z_1$	$x_1/2 \simeq y_1 \simeq 0{\cdot}25,\ z_1 \simeq 0$
rPD	$R\bar{3}m$–$R\bar{3}m(2c)$	3	1	$3(b)\ \bar{3}m\ 00\tfrac{1}{2}$	
				$9(d)\ .2/m\ \tfrac{1}{2}0\tfrac{1}{2}$	
$BC1$	$P6_322$–$P6_3$	9	1	$2(a)\ 32.\ 000$	
				$2(b)\ 3.2\ 00\tfrac{1}{4}$	
				$2(c)\ 3.2\ \tfrac{1}{3}\tfrac{2}{3}\tfrac{1}{4}$	
				$2(d)\ 3.2\ \tfrac{1}{3}\tfrac{2}{3}\tfrac{3}{4}$	
				$6(g)\ .2.\ x_100$	$x_1 \simeq 0{\cdot}40$
				$6(h)\ ..2\ x_2,2x_2,\tfrac{1}{4}$	$x_2 \simeq 0{\cdot}10$
				$12(i)\ 1\ x_3y_3z_3$	$x_3 \simeq 0{\cdot}60,\ y_3 \simeq z_3 \simeq 0{\cdot}10$
$BC2$	$P4_2/nnm$–$P4_2nm$	7	1	$4(e)\ ..2/m\ \tfrac{1}{4}\tfrac{1}{4}\tfrac{1}{4}$	
				$4(f)\ ..2/m\ \tfrac{3}{4}\tfrac{3}{4}\tfrac{3}{4}$	
				$8(j)\ .2.\ x_10\tfrac{1}{2}$	$x_1 \simeq 0{\cdot}30$
				$8(m)\ ..m\ x_2x_2z_2$	$x_2 \simeq z_2 \simeq 0{\cdot}70$
$ST2$	$P4_2/nbc$–$P4_2/n$	7	1	$8(h)\ .2.\ x_10\tfrac{1}{4}$	$x_1 \simeq 0{\cdot}35$
				$16(k)\ 1\ x_2y_2z_2$	$x_2 \simeq y_2 \simeq 0{\cdot}20,\ z_2 \simeq 0{\cdot}10$
$BC3$	$I422$–$I4$	6	2	$2(a)\ 422\ 000$	
				$2(b)\ 422\ 00\tfrac{1}{2}$	
			1	$8(i)\ .2.\ x_10\tfrac{1}{2}$	$x_1 \simeq 0{\cdot}35$
				$8(j)\ ..2\ x_2,\tfrac{1}{2}+x_2,\tfrac{1}{4}$	$x_2 \simeq 0{\cdot}30$
				$16(k)\ 1\ x_3y_3z_3$	$x_3 \simeq z_3 \simeq 0{\cdot}15,\ y_3 \simeq 0{\cdot}75$
$C(\pm Y)$	$Ia\bar{3}$–$Pa\bar{3}$	13	1	$8(b)\ .\bar{3}.\ \tfrac{1}{4}\tfrac{1}{4}\tfrac{1}{4}$	
				$16(c)\ .3.\ x_1x_1x_1$	$x_1 \simeq 0{\cdot}08$
				$24(d)\ 2..\ x_20\tfrac{1}{4}$	$x_2 \simeq 0{\cdot}30$
$\pm Y$	$Ia\bar{3}$–$Pa\bar{3}$	21	2	$24(d)\ 2..\ x_10\tfrac{1}{4}$	$x_1 \simeq 0$
			1	$8(b)\ .\bar{3}.\ \tfrac{1}{4}\tfrac{1}{4}\tfrac{1}{4}$	
				$24(d)\ 2..\ x_20\tfrac{1}{4}$	$x_2 \simeq 0{\cdot}30$
Y^*	$Ia\bar{3}d$–$I4_132$	3	1	$16(a)\ .\bar{3}.\ 000$	
oPa	$Immm$–$Pmmm$	3	1	$8(k)\ \bar{1}\ \tfrac{1}{4}\tfrac{1}{4}\tfrac{1}{4}$	

with $\beta = 1$, thus seemingly disproving the existence of the only example of flat points with $\beta = 3$ on three-periodic minimal surfaces. In the last two cases, the models are not accurate enough to permit a compelling decision.

References

FISCHER, W. & KOCH, E. (1987). *Z. Kristallogr.* **179**, 31–52.
FISCHER, W. & KOCH, E. (1989*a*). *Acta Cryst.* **A45**, 166–169.
FISCHER, W. & KOCH, E. (1989*b*). *Acta Cryst.* **A45**, 485–490.
FISCHER, W. & KOCH, E. (1989*c*). *Acta Cryst.* **A45**, 726–732.
HOPF, H. (1983). *Differential Geometry in the Large. Springer Lecture Notes in Mathematics*, No. 1000. Berlin: Springer.
HYDE, S. T. (1989). *Z. Kristallogr.* In the press.
KARCHER, H. (1988). *The Triply Periodic Minimal Surfaces of Alan Schoen and Their Constant Mean Curvature Companions.* Vorlesungsreihe No. 7. Sonderforschungsbereich 256. Institut für Angewandte Mathematik der Universität Bonn, Federal Republic of Germany.
KOCH, E. & FISCHER, W. (1988). *Z. Kristallogr.* **183**, 129–152.
KOCH, E. & FISCHER, W. (1989*a*). *Acta Cryst.* **A45**, 169–174.
KOCH, E. & FISCHER, W. (1989*b*). *Acta Cryst.* **A45**, 558–563.
LIDIN, S. & HYDE, S. T. (1987). *J. Phys.* **48**, 1585–1590.
NEOVIUS, E. R. (1883). *Bestimmung zweier spezieller periodischer Minimalflächen.* Helsingfors: Akad. Abhandlungen.
SCHOEN, A. H. (1970). *Infinite Periodic Minimal Surfaces Without Self-intersections. NASA Tech. Note* No. D-5541.
SCHWARZ, H. A. (1890). *Gesammelte mathematische Abhandlungen*, Vol. 1. Berlin: Springer.